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ORIGINAL ARTICLE

Modified solutions of some oscillators by iteration procedure

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Abstract The modified solutions of some nonlinear oscillators have been obtained based on the classical iteration procedure. In this article we have used the Fourier series and utilized all of its terms (sometimes approximately) in each iterative step. The third and fourth approximate frequencies of different nonlinear problems show good agreement with the exact values.

MATHEMATICS SUBJECT CLASSIFICATION: 34A34, 34B99

1. Introduction

The study of nonlinear problems is of crucial importance in the areas of Applied Mathematics, Physics and Engineering, as well as in other disciplines. Generally, the nonlinear problems are solved by converting them into linear equations imposing some conditions; but such linearization is not always possible. In this situation several methods are used to find approximate solutions to nonlinear problems, such as perturbation [1–4], harmonic balance (HB) [5–11], homotopy [12–15], iteration [16–26], etc. Among them the most widely used method is the perturbation method where the nonlinear term is small. Another technique (HB) is developed by Mickens [5,16,20] and farther work has been done by Lim [6,18], Hu [19,21–24], Wu [6,18] and so forth for handling the strong nonlinear problems. Recently, some authors [16–26] have used an iteration procedure which is valid for small as well as large amplitude of oscillation, to obtain the approximate frequency and the corresponding periodic solution of such nonlinear problems. Some authors [22,23,25,26] used modified version of this method to improve the results; but such modifications are not possible for the oscillators considered in this article. Fortunately the classical iteration procedure sometimes improves the results when the functions are not differentiable.

The main purpose of this article is to improve the solution in Ref. [13,14,26]. We have utilized the complete Fourier series (sometimes approximately) to expand the nonlinear terms in ‘Cosine series’. In certain cases the coefficients of Fourier series have been changed slightly to make it a standard form (of

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summation). Approximations from the first to the fourth (in a particular case the third approximation) have been presented and compared to the existing solutions.

2. The method

Let us consider a nonlinear oscillator modeled by

$$\ddot{x} + f(\ddot{x}, x) = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{1}$$

where over dots denote differentiation with respect to time, t. We choose the frequency Ω of this system. Then adding $\Omega^2 x$ to both sides of Eq. (1), we obtain

$$\ddot{x} + \Omega^2 x = \Omega^2 x - f(x, \ddot{x}) \equiv G(x, \ddot{x}). \tag{2}$$

Following [24], we formulate the iteration scheme as

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = G(x_k, \ddot{x}_k); \quad k = 0, 1, 2, \ldots, \tag{3}$$

together with

$$x_0(t) = A\cos(\Omega_0 t). \tag{4}$$

Herein x_{k+1} satisfies the conditions

$$x_{k+1}(0) = A, \quad \dot{x}_{k+1}(0) = 0. \tag{5}$$

At each stage of the iteration, Ω_k is determined by the requirement that secular terms (see [1] for details) should not occur in the solution. This procedure gives the sequence of solutions: $x_0(t)$, $x_1(t)$, ... The method can be proceed to any order of approximation; but due to growing algebraic complexity the solution is confined to a lower order usually the second [17].

3. Example

3.1. Anti-symmetric, piecewise constant force oscillator

Let us consider the anti-symmetric, piecewise constant force oscillator [26]

$$\ddot{x} = -sgn(x), \tag{6}$$

where

$$sgn(x) = \begin{cases} 1, & x > 0, \\ -1, & x < 0. \end{cases} \tag{7}$$

First we choose the case $x > 0$, therefore, Eq. (6) becomes

$$\ddot{x} + 1 = 0. \tag{8}$$

Obviously, Eq. (8) can be written as

$$\ddot{x} + \Omega^2 x = \Omega^2 x - 1. \tag{9}$$

Now the iteration scheme is (according to Eq. (3))

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = \Omega_k^2 x_k - 1. \tag{10}$$

Eq. (4) is rewritten as

$$x_0(t) = A\cos\theta, \tag{11}$$

where $\theta = \Omega t$. For $k = 0$, the Eq. (10) becomes

$$\ddot{x}_1 + \Omega_0^2 x_1 = \Omega_0^2 A\cos\theta - 1. \tag{12}$$

Expanding 1 in a Cosine series in interval $[0,\pi]$ and substituting in Eq. (12), we obtain

$$\ddot{x}_1 + \Omega_0^2 x_1 = \Omega_0^2 A\cos\theta - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos(2n-1)\theta}{2n-1}. \tag{13}$$

To avoid secular terms in the solution, we have to remove $\cos\theta$ from the right hand side of Eq. (13). Thus we have

$$\Omega_0^2 = \frac{4}{A\pi}. \tag{14}$$

Then solving Eq. (13) and satisfying the initial condition $x_1(0) = A$, we obtain

$$x_1(t) = A\left(\frac{1+\pi}{4}\cos\theta + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}\cos(2n-1)\theta}{(2n-1)((2n-1)^2-1)}\right). \tag{15}$$

This is the first approximate solution of Eq. (8) and the related Ω_1 is to be determined. The value of Ω_1 will be obtained from the solution of

$$\ddot{x}_2 + \Omega_1^2 x_2 = \Omega_1^2 x_1 - 1. \tag{16}$$

Substituting $x_1(t)$ from Eq. (15) into the right hand side of Eq. (16), we obtain

$$\begin{aligned}\ddot{x}_2 + \Omega_1^2 x_2 =& A\Omega^2\left(\frac{1+\pi}{4}\cos\theta + \sum_{n=2}^{\infty}\frac{(-1)^{n-1}\cos(2n-1)\theta}{(2n-1)((2n-1)^2-1)}\right) \\ &- \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos(2n-1)\theta}{2n-1}.\end{aligned} \tag{17}$$

Again avoiding secular terms in the solution of Eq. (17), we obtain

$$\Omega_1^2 = \frac{16}{A\pi(\pi+1)}. \tag{18}$$

Then solving Eq. (17) and satisfying initial condition, we obtain the second approximate solution,

$$\begin{aligned}x_2(t) =& A\left(\frac{1}{96}(12 + 12\pi + 5\pi^2)\cos\theta\right. \\ &- \sum_{n=2}^{\infty}\frac{(-1)^{n-1}\cos(2n-1)\theta}{(2n-1)((2n-1)^2-1)^2} \\ &\left.+ \frac{1+\pi}{4}\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\cos(2n-1)\theta}{(2n-1)((2n-1)^2-1)}\right).\end{aligned} \tag{19}$$

The third approximation x_3 and the value of Ω_2 will be obtained from the solution of

$$\ddot{x}_3 + \Omega_2^2 x_3 = \Omega_2^2 x_2 - 1. \tag{20}$$

Substituting x_2 from Eq. (19) into the right-hand side of Eq. (20), we have obtained the resulting expression as

$$\begin{aligned}\ddot{x}_3 + \Omega_2^2 x_3 =& A\Omega_2^2\left(\frac{1}{96}(12 + 12\pi + 5\pi^2)\cos\theta\right. \\ &- \sum_{n=2}^{\infty}\frac{(-1)^{n-1}\cos(2n-1)\theta}{(2n-1)((2n-1)^2-1)^2} + \frac{1+\pi}{4} \\ &\left.\times\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\cos(2n-1)\theta}{(2n-1)((2n-1)^2-1)}\right) \\ &- \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\cos(2n-1)\theta}{2n-1}.\end{aligned} \tag{21}$$

The value of Ω_2 will be obtained from the solution of Eq. (21), by using the condition of no secular terms in the solution. Thus we have

$$\Omega_2^2 = \frac{384}{A\pi(12 + 12\pi + 5\pi^2)}. \tag{22}$$

Then solving Eq. (21) and satisfying the initial condition, we obtain

$$\begin{aligned}x_3(t) =A\Bigg(&\frac{1}{384}(30+30\pi+15\pi^2+4\pi^3)\cos\theta \\ &+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\cos(2n-1)\theta}{(2n-1)((2n-1)^2-1)^3} \\ &-\frac{1+\pi}{4}\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\cos(2n-1)\theta}{(2n-1)((2n-1)^2-1)^2} \\ &+\frac{1}{96}(12+12\pi+5\pi^2)\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\cos(2n-1)\theta}{(2n-1)((2n-1)^2-1)}\Bigg)\end{aligned} \tag{23}$$

This is the third approximate solution of Eq. (8). After substituting this into the next and avoiding secular terms in the solution, we obtain

$$\Omega_3^2 = \frac{1536}{A\pi(30+30\pi+15\pi^2+4\pi^3)}. \tag{24}$$

Here the oscillator given by Eq. (6) is anti symmetric constant force oscillator. So, the values of Ω_0, Ω_1, Ω_2, $\Omega_3, \ldots$ will remain unchanged for next interval $[\pi, 2\pi]$ (where $x < 0$). Therefore, Ω_0, Ω_1, Ω_2, Ω_3, $\ldots$ respectively obtained by Eqs. 14,18, 22 and 24, ... represent the approximation of frequencies of oscillator (6).

3.2. Nonlinear singular oscillator

Let us consider a nonlinear singular oscillator

$$\ddot{x}+x^{-1}=0. \tag{25}$$

The Eq. (25) can be written as

$$\ddot{x}+\Omega^2x=\Omega^2x-x^{-1}. \tag{26}$$

According to Eq. (3), the iteration scheme of Eq. (26) will be

$$\ddot{x}_{k+1}+\Omega_k^2x_{k+1}=\Omega_k^2x_k-x_k^{-1}. \tag{27}$$

The first approximation $x_1(t)$ and the frequency Ω_0 will be obtained from the solution of (putting $k = 0$ in Eq. (27) and utilizing Eq. (4))

$$\ddot{x}_1+\Omega_0^2x_1=\Omega_0^2A\cos\theta-(A\cos\theta)^{-1}. \tag{28}$$

Now expanding $(\cos\theta)^{-1}$ in a Fourier Cosine series in interval $[0,\pi]$, the Eq. (28) reduces to

$$\ddot{x}_1+\Omega_0^2x_1=\Omega_0^2A\cos\theta-\frac{2}{A}\sum_{n=1}^{\infty}(-1)^{n-1}\cos(2n-1)\theta. \tag{29}$$

To check secular terms in the solution, we have to remove $\cos\theta$ from the right hand side of Eq. (29), and we obtain

$$\Omega_0^2=\frac{2}{A^2}. \tag{30}$$

Then solving Eq. (29) and satisfying the initial condition (according to Eq. (5)), we obtain

$$x_1(t)=A\left(1+\frac{1}{4}(-1+2\ln 2)\cos\theta-\sum_{n=2}^{\infty}\frac{(-1)^n}{4(n-1)n}\cos(2n-1)\theta\right). \tag{31}$$

This is the second approximation of Eq. (25) and the related Ω_1 is to be determined.

The second approximation $x_2(t)$ and the value of Ω_1 are obtained from the solution of

$$\ddot{x}_2+\Omega_1^2x_2=\Omega_1^2x_1-x_1^{-1}. \tag{32}$$

Substituting $x_1(t)$ from Eq. (31) into the right-hand side of Eq. (32), we obtain

$$\begin{aligned}\ddot{x}_2+\Omega_1^2x_2 =&A\Omega_1^2\Bigg((1+(-1+2\ln 2)/4)\cos\theta \\ &-\sum_{n=2}^{\infty}\frac{(-1)^n}{4(n-1)n}\cos(2n-1)\theta\Bigg) \\ &-\frac{1}{A}\sum_{n=1}^{\infty}(-1)^{n-1}a_{2n-1}\cos(2n-1)\theta,\end{aligned} \tag{33}$$

where

$$a_1=1.599611,\ a_3=0.983636,\ a_5=1.102235,\ a_7 =1.079400,\ a_9=1.083797,\ \ldots. \tag{34}$$

To avoid secular terms in the solution, we have to remove $\cos\theta$ from the right hand side of Eq. (33). Thus we have

$$\Omega_1^2=\frac{1.599611}{A^2(1+(-1+2\ln 2)/4)}. \tag{35}$$

Then Eq. (33) becomes,

$$\begin{aligned}\ddot{x}_2+\Omega_1^2x_2 =& -A\Omega_1^2\sum_{n=2}^{\infty}\frac{(-1)^n}{4(n-1)n}\cos(2n-1)\theta \\ &+\frac{1}{A}\sum_{n=2}^{\infty}(-1)^na_{2n-1}\cos(2n-1)\theta.\end{aligned} \tag{36}$$

The Eq. (36) approximately can be written as,

$$\begin{aligned}\ddot{x}_2+\Omega_1^2x_2 =& -A\Omega_1^2\sum_{n=2}^{\infty}\frac{(-1)^n}{4(n-1)n}\cos(2n-1)\theta \\ &+\frac{1.1}{A}\sum_{n=2}^{\infty}(-1)^n\cos(2n-1)\theta.\end{aligned} \tag{37}$$

Then solving Eq. (37) and satisfying the initial condition, we obtain the second approximation,

$$\begin{aligned}x_2(t) =&A((1-(3-4\ln 2)/16+1.1(-1+2\ln 2)/(4z))\cos\theta \\ &+\sum_{n=2}^{\infty}\left(\frac{(-1)^n}{(4(n-1)n)^2}+\frac{1.1(-1)^{n-1}}{4(n-1)n\Omega_1^2}\right)\cos(2n-1)\theta\Bigg),\end{aligned} \tag{38}$$

where

$$z=\frac{8}{(1+(-1+2\ln 2)/4)\sqrt{(3+\ln 2)(4+\ln 16)}}. \tag{39}$$

The third approximation x_3 and the value of Ω_2 are obtained from the solution of

$$\ddot{x}_3+\Omega_2^2x_3=\Omega_2^2x_2-x_2^{-1}. \tag{40}$$

Substituting $x_2(t)$ from Eq. (38) into the right-hand side of Eq. (40) and utilizing the same method, we obtain

$$\ddot{x}_3+\Omega_2^2x_3=\sum_{n=2}^{\infty}\left(A\Omega_2^2\left(\frac{(-1)^n}{(4(n-1)n)^2}+\frac{1.1(-1)^{n-1}}{4\Omega_1^2(n-1)n}\right)+(-1)^n\frac{1.26}{A}\right)\cos(2n-1)\theta, \tag{41}$$

where

$$\Omega_2^2=1.693744/(A^2(1-(3-4\ln 2)/16+1.1(-1+2\ln 2)/(4z))). \tag{42}$$

Then solving Eq. (41) and satisfying the initial condition, we obtain

$$x_3(t)=A\left(1.0672\cos\theta-\sum_{n=2}^{\infty}\left(\frac{1}{((n-1)n)^3}-\frac{1.1}{z((n-1)n)^2}+\frac{1.26}{z_1(n-1)n}\right)\cos\theta(2\cos 2\theta-1)\right), \tag{43}$$

where

$$z_1 = 1.693744/(1 - (3 - 4\ln 2)/16 + 1.1(-1 + 2\ln 2)/4z). \quad (44)$$

with the help of next iteration, we have

$$\Omega_3^2 = \frac{1.56636}{A^2}. \quad (45)$$

Therefore, Ω_0, Ω_1, Ω_2, Ω_3, ... respectively obtained by Eqs. 30,35,42 and 45, ... represent the approximation of frequencies of oscillator (25).

3.3. Nonrational restoring force oscillator

Now we may consider a nonrational restoring force oscillator

$$\ddot{x} + x^{1/3} = 0. \quad (46)$$

Obviously the Eq. (46) can be written as

$$\ddot{x} + \Omega^2 x = \Omega^2 x - x^{1/3}. \quad (47)$$

The iteration scheme is

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = \Omega_k^2 x_k - x_k^{1/3}. \quad (48)$$

Utilizing Eq. (4) and putting $k = 0$ in Eq. (48) the equation becomes,

$$\ddot{x}_1 + \Omega_0^2 x_1 = \Omega_0^2 A\cos\theta - (A\cos\theta)^{1/3}. \quad (49)$$

Using Fourier Cosine series, Eq. (49) reduce to

$$\ddot{x}_1 + \Omega_0^2 x_1 = \Omega_0^2 A\cos\theta - c_1\left(\cos\theta - \frac{3}{5}\left(\frac{1}{3}\cos 3\theta - \frac{1}{6}\cos 5\theta + \frac{7}{66}\cos 7\theta - \cdots\right)\right). \quad (50)$$

The Eq. (50) approximately can be written as

$$\ddot{x}_1 + \Omega_0^2 x_1 = \Omega_0^2 A\cos\theta - c_1\left(\cos\theta - \frac{3}{5}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n+1)}\cos(2n+1)\theta\right), \quad (51)$$

where

$$c_1 = \frac{3A^{1/3}\Gamma(7/6)}{\sqrt{\pi}\Gamma(2/3)}. \quad (52)$$

To avoid secular terms in the solution, we have to remove cosθ from the right hand side of Eq. (51). Thus we have

$$\Omega_0^2 = \frac{c_1}{A}. \quad (53)$$

Then solving Eq. (51) and satisfying initial condition (according to Eq. (5)), we obtain

$$x_1(t) = A(d_0 + d_2\cos 2\theta + d_4\cos 4\theta)\cos\theta, \quad (54)$$

where

$$d_0 = 1 + 1.2(\pi - 4\ln 2)/8, \quad d_2 = -1.2(3 - 2\ln 4)/4, \quad d_4 = 1.2(17 - 12\ln 4)/24. \quad (55)$$

Eq. (54) represents the first approximate solution of Eq. (46) and the related Ω_1 is to be determined. The value of Ω_1 will be obtained from the solution of

$$\ddot{x}_2 + \Omega_1^2 x_2 = \Omega_1^2 x_1 - x_1^{1/3}. \quad (56)$$

Substituting $x_1(t)$ from Eq. (54) into the right hand side of Eq. (56), the resulting expression is

$$\ddot{x}_2 + \Omega_1^2 x_2 = A\Omega_1^2(d_0 + d_2\cos 2\theta + d_4\cos 4\theta)\cos\theta - A^{1/3}\sum_{n=1}^{3} f_{2n-1}\cos(2n-1)\theta, \quad (57)$$

where

$$f_1 = 1.169869, \quad f_3 = -0.246939, \quad f_5 = 0.124850. \quad (58)$$

To avoid secular terms in the solution, we have to remove cosθ from the right hand side of Eq. (57). Then we have

$$\Omega_1^2 = f_1/A^{2/3}(d_0 + d_2/2), \quad (59)$$

Now solving Eq. (57) and satisfying the initial condition, we have the second approximation as

$$x_2(t) = A(g_0 + 2g_2\cos 2\theta)\cos\theta, \quad (60)$$

where

$$\begin{aligned} g_0 &= 1 + 2(d_2/2 + d_4/2 - f_3/e)(1/(5^2 - 1)), \\ g_2 &= -((d_2/2 + d_4/2 - f_3/e)(1/(3^2 - 1)) \\ &\quad -(d_4/2 - f_5/e)(1/(5^2 - 1))). \end{aligned} \quad (61)$$

The next approximation x_3 and the value of Ω_2 are obtained from the solution of

$$\ddot{x}_3 + \Omega_2^2 x_3 = \Omega_2^2 x_2 - x_2^{1/3}. \quad (62)$$

Substituting x_2 from Eq. (60) into the right-hand side of Eq. (62) and utilizing the condition of no secular terms, we obtain

$$\Omega_2^2 = 1.16866/(g_0 + g_2). \quad (63)$$

Thus, Ω_0, Ω_1, Ω_2, ... respectively obtained by Eqs. 53,59 and 63, ... represent the approximation of frequencies of oscillator (46).

4. Results and discussions

Only rearranging some nonlinear differential oscillators, iteration method [26] is utilized to approximate the solutions of those oscillators. This process significantly improves the results. First we consider the solution of Eq. (6). Here we have calculated Ω_0, Ω_1, Ω_2 and Ω_3. All the results are given in Table 1, to compare the approximate frequencies we have also given the existing results determined by Belendez [13].

Then we consider the solution of Eq. (25). Here we have calculated Ω_0, Ω_1, Ω_2 and Ω_3. All the results are given in Table 2, to compare the approximate frequencies we have also given the existing results determined by Mickens [26].

Recently, Mickens [11] has found approximate solutions of nonlinear singular oscillator Eq. (25) by both iteration and HB methods. He has shown that HB method sometimes measure better results than iteration method; but it is difficult to determine higher approximations (third or more than third) by HB method. Fortunately our method gives significantly better result than Mickens iteration formula. Sometimes it also measures better result than Mickens HB method (see Table 2; 9th and 10th column).

Finally we consider the solution of Eq. (46). Here we have calculated Ω_0, Ω_1 and Ω_2. All the results are given in Table 3. To compare the approximate frequencies we have also given the existing results determined by Belendez [14].

Table 1 Comparison of the approximate frequencies with exact frequency Ω_e [6] of $\ddot{x} + sgn(x) = 0$.

Amplitude	Ω_e	Ω_0 Er(%)	Ω_{B0} Er(%)	Ω_1 Er(%)	Ω_{B1} Er(%)	Ω_2 Er(%)	Ω_{B2} Er(%)	Ω_3 Er(%)
A	$\frac{1.110721}{A^{1/2}}$	$\frac{1.12838}{A^{1/2}}$ 1.59	$\frac{1.12838}{A^{1/2}}$ 1.59	$\frac{1.10892}{A^{1/2}}$ 0.16	$\frac{1.11035}{A^{1/2}}$ 0.65	$\frac{1.11089}{A^{1/2}}$ 0.02	$\frac{1.10803}{A^{1/2}}$ 0.24	$\frac{1.11071}{A^{1/2}}$ 0.001

Ω_0, Ω_1, Ω_2 and Ω_3 respectively denote the first, second, third and fourth modified approximate frequencies; Ω_{B0}, Ω_{B1} and Ω_{B2} respectively denote the first, second and third approximate frequencies obtained by Belendez [13]. Er(%) denotes percentage error.

Table 2 Comparison of the approximate frequencies with exact frequency Ω_e [26] of $\ddot{x} + x^{-1} = 0$.

Amplitude	Ω_e	Ω_0 Er(%)	Ω_{MI0} Er(%)	Ω_{MH0} Er(%)	Ω_1 Er(%)	Ω_{MI1} Er(%)	Ω_{MH1} Er(%)	Ω_2 Er(%)	Ω_{MH2} Er(%)	Ω_3 Er(%)
A	$\frac{1.253}{A}$	$\frac{1.414}{A}$ 12.84	$\frac{1.155}{A}$ 7.9	$\frac{1.414}{A}$ 12.84	$\frac{1.208}{A}$ 3.63	$\frac{1.018}{A}$ 18.1	$\frac{1.2728}{A}$ 1.6	$\frac{1.265}{A}$ 0.92	$\frac{1.2731}{A}$ 1.58	$\frac{1.252}{A}$ 0.14

Ω_0, Ω_1, Ω_2 and Ω_3 respectively denote the first, second, third and fourth modified approximate frequencies; Ω_{MI0} and Ω_{MI1} respectively denote the first and the second frequencies obtained by Mickens [26] iterative method; Ω_{MH0}, Ω_{MH1} and Ω_{MH2} respectively denote the first, second and third approximate frequencies obtained by Mickens [11] HB method. Er(%) denotes percentage error.

Table 3 Comparison of the approximate frequencies with exact frequency Ω_e [26] of $\ddot{x} + x^{1/3} = 0$.

Amplitude	Ω_e	Ω_0 Er(%)	Ω_{B0} Er(%)	Ω_1 Er(%)	Ω_{B1} Er(%)	Ω_2 Er(%)	Ω_{B2} Er(%)
A	$\frac{1.07045}{A^{1/3}}$	$\frac{1.07685}{A^{1/3}}$ 0.6	$\frac{1.07685}{A^{1/3}}$ 0.6	$\frac{1.07030}{A^{1/3}}$ 0.014	$\frac{1.06861}{A^{1/3}}$ 0.17	$\frac{1.07057}{A^{1/3}}$ 0.012	$\frac{1.07019}{A^{1/3}}$ 0.024

Ω_0, Ω_1 and Ω_2 respectively denote the first, second and third modified approximate frequencies; Ω_{B0}, Ω_{B1} and Ω_{B2} respectively denote the first, second and third approximate frequencies obtained by Belendez [14]. Er(%) denotes percentage error.

In most of the articles the results have been improved by modifying the method [23–25]; but in this article it has been shown that the results have been improved only by rearranging the governing equations of some oscillators. So, the modification is not only important; but rearrangement is also important in the case of iteration procedure.

5. Conclusion

An iteration technique has been applied simply rearranging some oscillators. The first to the fourth (in a particular case the third) approximate frequencies are better than corresponding frequencies which have been shown by other techniques. It can be observed that the third and the fourth approximation provide excellent results.

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